

UNIFORMITY AND FUNCTIONAL EQUATIONS FOR LOCAL ZETA FUNCTIONS OF \mathfrak{K} -SPLIT ALGEBRAIC GROUPS

BY MARK N. BERMAN

ABSTRACT. We study the local zeta functions of an algebraic group \mathcal{G} defined over \mathfrak{K} together with a faithful \mathfrak{K} -rational representation ρ for a finite extension \mathfrak{K} of \mathbb{Q} . These are given by integrals over \mathfrak{p} -adic points of \mathcal{G} determined by ρ for a prime \mathfrak{p} of \mathfrak{K} . We prove that the local zeta functions are almost uniform for all \mathfrak{K} -split groups whose unipotent radical satisfies a certain lifting property. This property is automatically satisfied if \mathcal{G} is reductive. We provide a further criterion in terms of invariants of \mathcal{G} and ρ which guarantees that the local zeta functions satisfy functional equations for almost all primes of \mathfrak{K} . We obtain these results by using a \mathfrak{p} -adic Bruhat decomposition of Iwahori and Matsumoto [IM] to express the zeta function as a weighted sum over the Weyl group W associated to \mathcal{G} of generating functions over lattice points of a polyhedral cone. The functional equation reflects an interplay between symmetries of the Weyl group and reciprocities of the combinatorial object. We construct families of groups with representations violating our second structural criterion whose local zeta functions do not satisfy functional equations. Our work generalizes results of Igusa [Igu] and du Sautoy and Lubotzky [dSL] and has implications for zeta functions of finitely generated torsion-free nilpotent groups.

1. INTRODUCTION

Let \mathcal{G} be an algebraic group defined over a finite extension \mathfrak{K} of \mathbb{Q} and let $\rho : \mathcal{G} \rightarrow \mathrm{GL}_n$ be a faithful \mathfrak{K} -rational representation of \mathcal{G} . Let \mathfrak{O} be the ring of integers of \mathfrak{K} and \mathfrak{p} a prime of \mathfrak{O} . We denote the localization of \mathfrak{K} at \mathfrak{p} by $\mathfrak{K}_{\mathfrak{p}}$, its ring of integers by \mathfrak{o} (the dependence on \mathfrak{p} being understood) and the size of its residue field by q . Let π be a fixed uniformizing parameter for \mathfrak{o} . For $X \leq \mathcal{G}$ put

$$\begin{aligned} X^+ &:= \{g \in X(\mathfrak{K}_{\mathfrak{p}}) \mid \rho(g) \in \mathrm{M}_n(\mathfrak{o})\}, \\ X(\mathfrak{o}) &:= \{g \in X(\mathfrak{K}_{\mathfrak{p}}) \mid \rho(g) \in \mathrm{GL}_n(\mathfrak{o})\}. \end{aligned}$$

We define the local zeta function of (\mathcal{G}, ρ) at the prime \mathfrak{p} to be

$$Z_{\mathcal{G}, \rho, \mathfrak{p}}(s) := \int_{\mathcal{G}^+} |\det \rho(g)|^s \mu_{\mathcal{G}}(g),$$

where s is a complex variable, $|\cdot|$ is the \mathfrak{p} -adic absolute value and $\mu_{\mathcal{G}}$ is the right invariant Haar measure on $\mathcal{G}(\mathfrak{K}_{\mathfrak{p}})$ normalized such that $\mu_{\mathcal{G}}(\mathcal{G}(\mathfrak{o})) = 1$.

The local zeta functions $Z_{\mathcal{G}, \rho, \mathfrak{p}}(s)$ are said to be (almost) uniform if there exists a rational function $Q(X, Y)$ such that for (almost) all primes \mathfrak{p} , $Z_{\mathcal{G}, \rho, \mathfrak{p}}(s) = Q(q, q^{-s})$.

2000 *Mathematics Subject Classification.* 05A15, 11S45, 20E07, 22E50.

Key words and phrases. Zeta functions of algebraic groups, zeta functions of finitely generated torsion-free nilpotent groups, enumerative combinatorics, p -adic integration, uniformity, local functional equations.

In this case, set $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)|_{q \rightarrow q^{-1}} := Q(q^{-1}, q^s)$. We say that $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ satisfies a functional equation if

$$(1.1) \quad Z_{\mathcal{G},\rho,\mathfrak{p}}(s)|_{q \rightarrow q^{-1}} = (-1)^m q^{a+bs} Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$$

for some integers m, a, b . In fact there is a well-defined notion of a functional equation even in cases in which the local zeta functions attached to an object are not almost uniform but rather are determined by counting rational points on certain varieties (see, for instance, [Vol]). We will impose conditions on (\mathcal{G}, ρ) that will imply that the local zeta functions are almost uniform. However, the question of uniformity for these integrals, in general, remains open.

Early interest in the function $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ came from the fact that it is a natural generalization of the Dedekind zeta function. Indeed, the latter may be obtained by taking $\mathcal{G} = \mathrm{GL}_1$ and ρ the natural representation. It was studied in other special cases by Hey, Weil, Tamagawa, Satake and Macdonald [Hey, Wei, Tam, Sat, Mac]. Tamagawa considered the case $\mathcal{G} = \mathrm{GL}_n$ with the natural representation and showed that the global zeta function (defined to be the product of the local zeta functions over all primes \mathfrak{p}) has meromorphic continuation to the whole of the complex plane. In [dSW, Chapter 6] the authors proved that for several families of classical groups, the global zeta functions have natural boundaries and thus cannot be meromorphically continued. Nevertheless, it remains interesting to ask which properties of the Dedekind zeta function carry over to the function $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ for all groups \mathcal{G} and representations ρ .

The question of whether the functions $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ are (almost) uniform and satisfy functional equations was first addressed in a more general setting by Igusa [Igu]. He chose Serre's canonical measure on \mathcal{G} , which differs from the Haar measure used in our definition. He was able to derive an explicit form for $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ in terms of q and q^{-s} . This form involved a certain sum, over the Weyl group W of \mathcal{G} , of rational functions weighted by the length function on W . Igusa was able to utilize a symmetry of the Weyl group together with reciprocities for the rational functions to prove that the local zeta functions satisfy functional equations for almost all primes. The main tool he employed was a \mathfrak{p} -adic Bruhat decomposition due to Iwahori and Matsumoto [IM]. Although his integral differed from our ours, the method he employed will be essential to our analysis.

Independent interest in the function $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ was generated by [GSS], in which the authors showed that the zeta function in fact expresses a subgroup counting function in a completely different context. Let Γ be a finitely generated torsion-free nilpotent group (or \mathcal{T} -group) and for $*$ $\in \{\leq, \triangleleft, \wedge\}$ define

$$\zeta_{\Gamma}^*(s) := \sum_{n=1}^{\infty} a_n^* n^{-s},$$

where $a_n^{\leq}, a_n^{\triangleleft}$ and a_n^{\wedge} denote the number of subgroups H of Γ of index n satisfying $H \leq \Gamma$, $H \triangleleft \Gamma$ and $\widehat{H} \cong \widehat{\Gamma}$ respectively. Here $\widehat{}$ denotes the profinite completion. $\zeta_{\Gamma}^{\wedge}(s)$ has been given no name to date; we refer to it as the proisomorphic zeta function of Γ . Define local zeta functions for each prime p by

$$\zeta_{\Gamma,p}^*(s) := \sum_{k=0}^{\infty} a_{p^k}^* p^{-ks}.$$

Grunewald, Segal and Smith showed that for $* \in \{\leq, \triangleleft, \wedge\}$ and for each prime p , $\zeta_{\Gamma,p}^*(s)$ is a rational function in p^{-s} . They also showed that, as a straightforward consequence of nilpotency, the zeta function decomposes as an Euler product:

$$\zeta_{\Gamma}^*(s) = \prod_p \zeta_{\Gamma,p}^*(s).$$

Furthermore, they realized the local proisomorphic zeta function $\zeta_{\Gamma,p}^{\wedge}(s)$ as the local zeta function $Z_{\mathcal{G},\rho,p}(s)$ of an algebraic group with an associated \mathbb{Q} -rational representation (hence all such realizations are *a fortiori* rational functions in p^{-s}). This prompted du Sautoy and Lubotzky to study $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ with a view to ascertaining whether the local proisomorphic zeta functions $\zeta_{\Gamma,p}^{\wedge}(s)$ would be uniform and whether they would satisfy functional equations.

It is well-known that if \mathcal{G}_0 is the connected component of \mathcal{G} then \mathcal{G}_0 can be expressed as $N \rtimes G$, where N is the unipotent radical of \mathcal{G}_0 and G is a (connected) reductive subgroup (see, for instance, [Bor, p. 9]). In [dSL] du Sautoy and Lubotzky made a reduction to an integral over the subgroup G and showed that the zeta function $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ would be unchanged for almost all primes \mathfrak{p} . Specifically, they showed that

$$(1.2) \quad Z_{\mathcal{G},\rho,\mathfrak{p}}(s) = \int_{G^+} |\det \rho(g)|^s \theta(g) \mu_G(g),$$

where θ is the function $G \rightarrow \mathbb{R}_{\geq 0}$ given by

$$(1.3) \quad \theta(g) := \mu_N(\{n \in N \mid ng \in \mathcal{G}_0^+\})$$

and μ_N is the right invariant Haar measure on N normalized such that $\mu_N(N(\mathfrak{o})) = 1$. They were also able to decompose the function θ into pieces defined relative to a normal series for N ; this relied on a certain lifting assumption on \mathcal{G}_0 (see [dSL, Assumption 2.3], restated in our paper as Condition 3.5). They further assumed that G would split over \mathfrak{K} , that θ would be (the \mathfrak{p} -adic absolute value of) a character on G , that the rank of the maximal central torus of G would be 1, and that among the irreducible components ρ_1, \dots, ρ_r of $\rho|_G$ there would be one whose dominant weight ‘dominates’ the dominant weights of the other components. We refer the reader to [dSL] for a definition of the latter. Under these assumptions, they were able to extend Igusa’s method to obtain an explicit form for the local zeta functions $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$, deducing that they would be almost uniform in \mathfrak{p} and would satisfy functional equations.

We now state our main theorem.

Theorem 1.1. *Let \mathcal{G} be an algebraic group defined over \mathfrak{K} and let ρ be a faithful \mathfrak{K} -rational representation $\mathcal{G} \rightarrow \mathrm{GL}_n$. Write $\mathcal{G}_0 = N \rtimes G$, where \mathcal{G}_0 is the connected component of \mathcal{G} , N is its unipotent radical and G is a connected reductive subgroup. Suppose that, after fixing some suitable decomposition of the representation space, (\mathcal{G}_0, G, ρ) satisfies Condition 3.5 (cf. Section 3) for almost all primes \mathfrak{p} . Suppose further that G splits over \mathfrak{K} . Let d denote the rank of the maximal central torus of \mathcal{G}_0 and r the number of irreducible components of $\rho|_G$. Then*

- (1) *the local zeta functions $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ are almost uniform, i.e. uniform outside a finite set of primes*
- (2) *if $r = d$, for almost all primes \mathfrak{p} , $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ satisfies the functional equation*

$$Z_{\mathcal{G},\rho,\mathfrak{p}}(s)|_{q \rightarrow q^{-1}} = (-1)^{l+d} q^{|\Phi^+| + c - ns} Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$$

where l is the number of fundamental roots of the root system associated to G , Φ^+ is a set of positive roots, and c is a non-negative integer. If \mathcal{G} is reductive then $c = 0$.

For a faithful representation we necessarily have $r \geq d$ (each irreducible component has a maximal central torus of rank 1 or 0; see [Igu, p. 700]). We will show in Section 6 that in one direction our result is best possible in the following sense: there exist groups \mathcal{G} and representations ρ for which $r > d$ whose local zeta functions do not satisfy functional equations in the sense of (1.1). A more complicated counterexample has in fact been given previously by Martin [Mar]. He studied the integral as defined by Igusa for a certain 3402-dimensional irreducible representation of GL_7 . Here, since $r = d = 1$, we infer from our Theorem 1.1 that it is the different choice of measure in this example (namely the canonical measure) that is responsible for the break-down of the functional equation.

The proof of Theorem 1.1 relies on the splitting assumption to utilize a \mathfrak{p} -adic Bruhat decomposition of Iwahori-Matsumoto. As suggested in [dSL, p. 73], it may be possible to remove this assumption using the notion of a \mathfrak{p} -adic decomposition for non-split reductive groups due to Bruhat and Tits (see [BT1] and [BT2]). Condition 3.5 is needed to reduce to the case of a reductive group, and cannot easily be removed without a deeper understanding of the action of the reductive subgroup on the unipotent radical, as described below. In the case that \mathcal{G} is reductive, Condition 3.5 holds trivially. We thus obtain the following.

Corollary 1.2. *If \mathcal{G} is a \mathfrak{K} -split reductive group defined over \mathfrak{K} with faithful \mathfrak{K} -rational representation ρ then its local zeta functions satisfy statements (1) and (2) of Theorem 1.1.*

Our theorem generalizes [dSL, Theorem 6.1] as follows. We allow the maximal central torus to have arbitrary rank and we no longer make the assumption that the function θ is (the \mathfrak{p} -adic absolute value of) a character on G . We do not assume that there is any relationship between the dominant weights of the irreducible components of $\rho|_G$; rather, we restrict the number of components to d , the rank of the maximal central torus of G . In [dSL], the authors proved a functional equation in the case that $d = 1$, r is arbitrary and among the dominant weights of the irreducible components of the representation, there is one which ‘dominates’ all the others. Note that our Theorem 1.1 (2) does not reduce to their Theorem A (2) under their assumptions. In fact their proof of the latter is flawed, which we will explain at the end of Section 5.2. However, in the case $r = 1$ their proof is valid and their result becomes a special case of our Theorem 1.1 (2).

The functional equation in part (2) of our main theorem comes, as in Igusa’s setting, from an interplay between symmetries of the Weyl group associated to the algebraic group, and reciprocities of the generating functions in the weighted sum. However, in our case these generating functions are not just geometric series (as in [Igu] and [dSL]); rather, they are certain generalized generating functions over subsets of a polyhedral cone. The reciprocity property we use to prove the functional equation is an extension of a reciprocity theorem due to Stanley [Sta, Theorem 4.6.14]. Our proof relies crucially on the fact that the cone is simple. By this we mean that the set of lattice points it contains is freely generated as a commutative monoid (note that a simple cone is, in particular, simplicial). The simplicity of the cone is in turn a consequence of the condition $r = d$. Indeed, it

is this insight that enables us to construct our counterexamples in Section 6. An important feature of our combinatorial data is the existence of a ‘minimal vector’ in Corollary 2.7 below, in a similar vein to [Sta, Corollary 4.6.16], although our case is slightly different due to the presence of internal structure in the cone. Our proof of a functional equation is reminiscent of [Vol], where the generating functions varied in different cells defined by linear forms (see [Vol, Proposition 2.1]). In [KV, Theorem A], the authors considered sums of generating functions possessing ‘inversion properties’ analogous to Lemma 2.5 in the present paper, weighted by the numbers of non-degenerate flags in a finite formed space. While the settings are all different, they share a common feature in utilizing both combinatorial reciprocity properties and symmetries of a Weyl or Coxeter group to prove a functional equation (as is the case also in [Igu] and [dSL]).

We now explain why we have included Condition 3.5 in Theorem 1.1 and how this relates to [dSL, Theorem 6.1]. In [dSL], the authors analyzed the automorphism group of U_4^0 , the Lie algebra of upper triangular 4 by 4 matrices, together with a natural representation, and showed that in this case θ (defined in (1.3)) is only a ‘piecewise’ character. Specifically, they were able to divide the domain of integration into two regions such that θ was a character on each. In Section 3 we will make this notion of ‘piecewise characters’ more precise and show that θ will be a piecewise character provided that (\mathcal{G}_0, ρ) satisfies Condition 3.5. We need this property of θ in order to carry out our analysis of the reduced integral expression for $Z_{\mathcal{G}, \rho, \mathfrak{p}}(s)$ given in (1.2). Since the condition is somewhat technical, we postpone a proper description of it until Section 3. Roughly speaking, the condition states that quotients of \mathcal{G}_0 by certain normal subgroups N_i of the unipotent radical, together with natural representations $\varphi_i : \mathcal{G}_0/N_i \rightarrow \mathrm{GL}_n$, enjoy the property that integral points of \mathcal{G}_0/N_i defined with respect to φ_i lift to integral points of \mathcal{G}_0 with respect to ρ . Corollary 4.5 in [dSL] states that our Condition 3.5 is satisfied for almost all \mathfrak{p} . This is in fact incorrect. For instance, if \mathcal{G} is taken to be the automorphism group of U_5^0 and ρ the representation with respect to the standard basis for U_5^0 , it can be checked by a straightforward calculation that Condition 3.5 fails to be satisfied for all primes \mathfrak{p} (see [Ber, pp. 78-84]). Therefore, a correct formulation of [dSL, Theorem 6.1] requires Condition 3.5 just as our Theorem 1.1 does.

Motivated by the examples presented in Section 6, we ask the following.

Question 1.3. *Does there exist a finitely generated torsion-free nilpotent group Γ such that, for almost all primes, the local zeta function $\zeta_{\Gamma, \mathfrak{p}}^\wedge(s)$ does not satisfy a functional equation?*

To explain why this might be so, we briefly describe the relationship between zeta functions of algebraic groups and proisomorphic zeta functions of \mathcal{T} -groups. Consider a nilpotent Lie ring L over \mathbb{Z} of finite rank n . Put $L_p = L \otimes \mathbb{Z}_p$ and $\mathcal{L} = L \otimes \mathbb{Q}_p$. If H is both a Lie subring and a full \mathbb{Z}_p -sublattice of L_p (written $H \leq L_p$), we write $H \cong L_p$ if there exists $g \in \mathrm{Aut} \mathcal{L}$ such that $L_p \cdot g = H$. The following two results connect zeta functions of groups with zeta functions of algebraic groups:

Proposition 1.4 (cf. [GSS, Theorem 4.1]). *Given a \mathcal{T} -group Γ , there exists a nilpotent Lie ring L of finite rank over \mathbb{Z} such that for almost all primes p ,*

$$\zeta_{\Gamma, \mathfrak{p}}^\wedge(s) = \zeta_{L, \mathfrak{p}}^\wedge(s) := \sum_{\substack{H \leq L_p \\ H \cong L_p}} |L_p : H|^{-s}.$$

Proposition 1.5 (cf. [GSS, Proposition 4.2]). *Given a Lie ring L of finite rank over \mathbb{Z} , let ρ be the representation $\text{Aut} \mathcal{L} \rightarrow \text{GL}_n(\mathbb{Q}_p)$ defined by fixing some \mathbb{Z} -basis for L . Then $\text{Aut} \mathcal{L} = \mathcal{G}(\mathbb{Q}_p)$, where \mathcal{G} is the algebraic automorphism group of $L \otimes \mathbb{Q}$, and for each prime p ,*

$$\zeta_{L,p}^\wedge(s) = \int_{\mathcal{G}^+} |\det \rho(g)|^s \mu_{\mathcal{G}}(g).$$

Incidentally, Proposition 1.5 is of independent interest as it generalizes to counting isomorphic subrings in Lie rings $L \otimes \mathfrak{o}$, in which case the results of the present paper apply. A result of Bryant and Groves [BG, Theorem A] implies that every algebraic group defined over \mathbb{Q} together with every possible faithful \mathbb{Q} -rational representation can be realized as the quotient of the automorphism group of some nilpotent Lie algebra by the group of IA-automorphisms (these are the automorphisms acting trivially on the abelianisation of the Lie algebra), together with a natural representation. Unfortunately this does not give an immediate answer to Question 1.3. Even in cases where the group of IA-automorphisms coincides with the unipotent radical, we do not yet have sufficient understanding of the effect of the latter on the integral to extend our counterexamples to proisomorphic zeta functions of \mathcal{T} -groups using the approach of [BG].

Our results contribute to a broader picture of zeta functions of \mathcal{T} -groups. Local functional equations are known to be satisfied for all subgroup counting zeta functions [Vol] and for normal subgroup counting zeta functions of groups of class at most 2 [Vol, Paa], while counterexamples are known in the normal subgroup case already in class 3 [dSW]. On the other hand, there are examples of \mathcal{T} -groups whose local zeta functions of both subgroup and normal subgroup type are non-uniform [dS]. These examples corroborated an analysis of the zeta functions $\zeta_{\Gamma,p}^*(s)$ ($*$ $\in \{\leq, \triangleleft\}$) in [dSG] which established a theoretical link between counting subgroups of a \mathcal{T} -group Γ and counting \mathbb{F}_p -points on an algebraic variety associated to Γ . However, this analysis was not carried out for proisomorphic zeta functions, and it remains to be seen whether they too can exhibit non-uniform behaviour. In view of the dichotomy between $\zeta_{\Gamma,p}^\leq(s)$ and $\zeta_{\Gamma,p}^\triangleleft(s)$, it is an interesting open question whether proisomorphic zeta functions satisfy local functional equations in general. If Question 1.3 is answered in the affirmative, it will be of great interest to characterize those \mathcal{T} -groups whose local proisomorphic zeta functions do not satisfy functional equations. On the other hand, Theorem 1.1 extends the known classes of algebraic groups and representations for which the local zeta functions are uniform and do satisfy functional equations. In view of Propositions 1.4 and 1.5, this implies corresponding results for a larger class of \mathcal{T} -groups than previously known (cf. [dSL, Theorem B]).

The paper is laid out as follows. In Section 2 we consider generating functions over polyhedral cones similar to those considered by Stanley [Sta]. The chief difference is that in our context the summand varies within a finite number of cells into which the cone is subdivided. We combine a new combinatorial result with techniques of Stanley [Sta] and Igusa [Igu] to show that if the cone is simple then a certain sum of generating functions weighted by elements of an abstract Weyl group satisfies a reciprocity property. In Section 3 we explain how the work of du Sautoy and Lubotzky [dSL] can be extended to obtain more delicate control over the form of the reduced integral expression for $Z_{\mathcal{G},\rho,\mathfrak{p}}(s)$ obtained by restricting the domain of integration to a connected reductive subgroup. In Section 4 we use a \mathfrak{p} -adic

Bruhat decomposition due to Iwahori and Matsumoto to obtain a combinatorial expression for the zeta function as a sum of generating functions over a polyhedral cone, utilizing methods of [Igu] and [dSL]. We are then able to deduce Theorem 1.1 in Section 5, utilizing the results of Section 2. In Section 6 we give examples of local zeta functions of algebraic groups that do not satisfy functional equations.

Acknowledgements. I wish to thank Marcus du Sautoy, my doctoral supervisor, for many inspiring conversations. I am also indebted to Nir Avni, Benjamin Klopsch, Uri Onn, Pirita Paaanen, Ilya Tyomkin and Christopher Voll for valuable discussions and suggestions. In particular, I wish to acknowledge Nir Avni for contributing to the construction of the examples in Section 6 and to the proof of Lemma 3.10. I am grateful to my father, Peter Berman, and to Christopher Voll for their feedback on the presentation. During the course of this work I obtained support from several sources. I wish to thank the Rhodes Trust for a scholarship, Alex Lubotzky for a postdoctoral grant, and the Lady Davis Fellowship Trust for a Golda Meir postdoctoral fellowship. For financial support, I am grateful to Merton College and the Institute of Mathematics, University of Oxford, and to the Einstein Institute of Mathematics, The Hebrew University of Jerusalem. I also thank all of them for their hospitality. Finally, I wish to thank the referees for their insightful and helpful remarks.

We use the following notation:

\mathbb{N}	the set of positive integers
\mathbb{N}_0	the set of non-negative integers
\mathbb{Q}_p	the set of p -adic numbers
\mathbb{Z}_p	the set of p -adic integers
$[n]$	the set $\{1, \dots, n\}$
$\text{Sym}(n)$	the symmetric group on $\{1, \dots, n\}$
$ \cdot $	the \mathfrak{p} -adic absolute value
$v(x)$	the valuation of x , for $x \in \mathfrak{K}_{\mathfrak{p}}$

2. RECIPROCITIES OF GENERATING FUNCTIONS OVER CONES

We begin by fixing our terminology, most of which is standard. The dimension of a (non-empty) subset of \mathbb{R}^m is the dimension of the subspace spanned by it. A (linear) hyperplane H is a set of the form $\{v \in \mathbb{R}^m \mid v \cdot w = 0\}$, where w is some fixed vector in \mathbb{R}^m . We will always assume such a vector has been fixed even when this is not made explicit. We set $H^> := \{v \in \mathbb{R}^m \mid v \cdot w > 0\}$ and similarly for H^{\geq} , $H^<$, H^{\leq} , and we write $H^0 = H$. We call a hyperplane rational if the vector defining it has rational coordinates with respect to the standard basis for \mathbb{R}^m . We call a subset of \mathbb{R}^m a cone if it is an intersection of closed half-spaces; that is, sets of the form H_i^{\geq} for some hyperplanes H_i . A cone is pointed if it contains no lines and it is rational if it may be defined by rational hyperplanes. A cone is simplicial if it contains a finite subset S such that every point of the cone is uniquely expressible as a non-negative \mathbb{R} -linear combination of elements of S . It is simple if it is pointed and the set of lattice points contained in it is freely generated as a commutative monoid (in particular, a simple cone is simplicial). Finally, we define a polyhedral cell complex. This is a cone \mathcal{C} and a family \mathcal{F} of cells defined by two finite collections of hyperplanes – bounding hyperplanes $\{B_i\}_{i \in M}$ and internal

hyperplanes $\{H_i\}_{i \in K}$. The cone of the complex is defined to be

$$\mathcal{C} := \bigcap_{i \in M} B_i^>.$$

The cells are defined to be sets of the form

$$\bigcap_{i \in M} B_i^{s_i} \cap \bigcap_{j \in K} H_j^{t_j}$$

where each $s_i \in \{0, >\}$ and each $t_j \in \{0, <, >\}$. Thus every cell is open in its support and is contained in \mathcal{C} ; also, \mathcal{C} is a disjoint union of the cells. For a subset X of \mathbb{R}^m , let \overline{X} denote its closure with respect to the standard Euclidean metric. Given two cells $F_1, F_2 \in \mathcal{F}$, we define F_1 to be a face of F_2 , written $F_1 \leq F_2$, if $\overline{F_1} \subseteq \overline{F_2}$. For each $I \subseteq M$ set

$$\mathcal{C}_I := \bigcap_{i \in I} B_i^> \cap \bigcap_{j \in M \setminus I} B_j^>.$$

For instance, $\mathcal{C}_\emptyset = \mathcal{C}$ and $\mathcal{C}_M = \text{Int}(\mathcal{C})$ (the interior taken with respect to the support of \mathcal{C}). Set

$$\mathcal{F}_I := \{F \in \mathcal{F} \mid F \subseteq \mathcal{C}_I\}.$$

If $e \in \mathcal{C}$, denote by F_e the unique cell in the complex containing e .

Definition 2.1. We call a function $\gamma : \mathcal{F} \rightarrow \mathbb{Z}^m$ *piecewise constant* on the complex if for each cell F there exists $C_F \in \mathbb{Z}^m$ such that $C_F \cdot e = \gamma(F_e) \cdot e$ for all $e \in \overline{F} \cap \mathbb{Z}^m$.

Definition 2.2. For $Y \subseteq \mathcal{C}$, $\gamma : \mathcal{F} \rightarrow \mathbb{Z}^m$, $A, B \in \mathbb{Z}^m$, q a prime power and s a complex variable put

$$E_{Y,A,B,\gamma}(q, q^{-s}) := \sum_{e \in Y \cap \mathbb{Z}^m} q^{(A+\gamma(F_e)) \cdot e - (B \cdot e)s}.$$

Usually we will simply write this as $E_Y(q, q^{-s})$. We can now state our main combinatorial result:

Theorem 2.3. *Let $(\mathcal{C}, \mathcal{F})$ be a polyhedral cell complex in \mathbb{R}^m defined by rational hyperplanes, and suppose \mathcal{C} is a simplicial cone defined by m bounding hyperplanes with $\dim \mathcal{C} = m$. Let $\gamma : \mathcal{F} \rightarrow \mathbb{Z}^m$ be a piecewise constant function on the complex and let $A, B \in \mathbb{Z}^m$. Suppose that $B \cdot e > 0$ for all $0 \neq e \in \mathcal{C}$. Then for each $I \subseteq [m]$, $E_{\mathcal{C}_I}(q, q^{-s})$ is a rational function in q, q^{-s} . Furthermore,*

$$(2.1) \quad E_{\mathcal{C}_I}(q, q^{-s})|_{q \rightarrow q^{-1}} = (-1)^m E_{\mathcal{C}_{[m] \setminus I}}(q, q^{-s}).$$

Note that Theorem 2.3 reduces (essentially) to a special case of [Sta, Theorem 4.6.14] if $I = \emptyset$ and there are no internal hyperplanes. On the other hand, if $I \notin \{\emptyset, [m]\}$, the reciprocity result is best possible in the sense that if \mathcal{C} is not simplicial, (2.1) will not necessarily hold (it is easy to construct examples).

The proof of the theorem depends on the following two results. The first of these is new, as far as we are aware, while the second is essentially a restatement of [Sta, Theorem 4.6.14] in a form suitable to our context.

Proposition 2.4. *Take $(\mathcal{C}, \mathcal{F})$ as in the hypothesis of Theorem 2.3. Then for each $F_0 \in \mathcal{F}$ and $I \subseteq [m]$ we have*

$$\sum_{\substack{F \in \mathcal{F}_I \\ F \geq F_0}} (-1)^{\dim F} = \begin{cases} 0 & \text{if } F_0 \notin \mathcal{F}_{[m] \setminus I} \\ (-1)^m & \text{if } F_0 \in \mathcal{F}_{[m] \setminus I}. \end{cases}$$

Proof. We prove this by induction on the number $|K|$ of internal hyperplanes. We start with the case $|K| = 0$. If $F_0 \in \mathcal{F}_{[m] \setminus I}$ then $\{F \in \mathcal{F}_I \mid F_0 \leq F\} = \mathcal{F}_{[m]}$, which consists of a single cell of dimension m , so we obtain the required expression. If $F_0 \notin \mathcal{F}_{[m] \setminus I}$, put $I_0 = \max\{J \subseteq [m] \setminus I \mid F_0 \subseteq \mathcal{C}_J\}$ and note that $\emptyset \subseteq I_0 \subsetneq [m] \setminus I$. Then, since \mathcal{C} is simplicial, we have

$$\begin{aligned} \sum_{\substack{F \in \mathcal{F}_I \\ F \geq F_0}} (-1)^{\dim F} &= \sum_{J \subseteq [m] \setminus (I \cup I_0)} (-1)^{m - |J|} \\ &= 0. \end{aligned}$$

Suppose now that $(\mathcal{C}, \mathcal{F})$ is a cell complex with internal hyperplanes $\{H_i\}_{i \in K}$. Let H be a hyperplane and let $(\mathcal{C}, \mathcal{F}^2)$ be the cell complex formed by \mathcal{C} with internal hyperplanes $\{H_i\}_{i \in K} \cup \{H\}$. For each $J \subseteq [m]$ define \mathcal{F}_J (respectively, \mathcal{F}_J^2) as before. Inductively, we assume that the result holds for $(\mathcal{C}, \mathcal{F})$. Note that \mathcal{F}^2 consists of all non-empty cells of the form $F \cap H^*$ where $F \in \mathcal{F}$ and $*$ $\in \{0, <, >\}$. Furthermore, for each $*$ $\in \{0, <, >\}$, $F \cap H^* \in \mathcal{F}_{[m] \setminus I}^2$ if and only if $F \in \mathcal{F}_{[m] \setminus I}$ and $F \cap H^* \neq \emptyset$. Thus it is sufficient to show that if $F_0 \in \mathcal{F}$, $*$ $\in \{0, <, >\}$ and $F_0 \cap H^* \neq \emptyset$, the following holds:

$$(2.2) \quad \sum_{\substack{F \in \mathcal{F}_I \\ F_0 \leq F}} (-1)^{\dim F} = \sum_{\substack{F' \in \mathcal{F}_I^2 \\ F_0 \cap H^* \leq F'}} (-1)^{\dim F'}.$$

If $F_0 \cap H^< \neq \emptyset$ then for each $F \geq F_0$ in \mathcal{F}_I we have $F \cap H^< \neq \emptyset$ and hence $\dim F = \dim F \cap H^<$. Thus

$$\{F' \in \mathcal{F}_I^2 \mid F' \geq F_0 \cap H^<\} = \{F \cap H^< \mid F \geq F_0, F \in \mathcal{F}_I\}$$

and (2.2) follows in this case. The case $F_0 \cap H^> \neq \emptyset$ is similar. Suppose that $F_0 \cap H \neq \emptyset$. If $F' \geq F_0 \cap H$ in \mathcal{F}_I^2 then $F' = F \cap H^*$ for some $F \geq F_0$ in \mathcal{F}_I and $*$ $\in \{0, <, >\}$. In order to establish (2.2) it is sufficient to show that

$$(2.3) \quad \sum_{\substack{* \in \{0, <, >\} \\ F \cap H^* \neq \emptyset}} (-1)^{\dim F \cap H^*} = (-1)^{\dim F}.$$

Suppose that there exists $*$ $\in \{0, <, >\}$ such that $F \subseteq H^*$. Then (2.3) follows immediately. The alternative possibility is that $F \cap H \neq \emptyset$ and $F \not\subseteq H$. In this case $F \cap H^> \neq \emptyset$ and $F \cap H^< \neq \emptyset$. We have $\dim F \cap H^> = \dim F \cap H^< = \dim(F \cap H) + 1 = \dim F$, and (2.3) follows. \square

Lemma 2.5. *Take $(\mathcal{C}, \mathcal{F})$, A , B and γ as in the hypothesis of Theorem 2.3. Then for each $F_0 \in \mathcal{F}$, $E_{F_0}(q, q^{-s})$ is a rational function in q and q^{-s} . Furthermore,*

$$(2.4) \quad E_{F_0}(q, q^{-s})|_{q \rightarrow q^{-1}} = (-1)^{\dim F_0} \sum_{F \leq F_0} E_F(q, q^{-s}).$$

Proof. Since γ is piecewise constant, there exists $C_{F_0} \in \mathbb{Z}^m$ such that $C_{F_0} \cdot e = \gamma(F_e) \cdot e$ for all $e \in \overline{F_0} \cap \mathbb{Z}^m$. We begin by proving the result in the case that $\overline{F_0}$ is simplicial. In this case, there exist $u_1, \dots, u_k \in \mathbb{Z}^m$ (where $k = \dim F_0$) such that each element of F_0 is uniquely expressible as a positive real linear combination of these k vectors. Denote by D_1 the set $\mathbb{Z}^m \cap \sum_{i=1}^k (0, 1] u_i$ and by D_0 the set $\mathbb{Z}^m \cap \sum_{i=1}^k [0, 1) u_i$. Then

$$\mathbb{Z}^m \cap F_0 = \prod_{\substack{e \in D_1 \\ t \in \mathbb{N}_0^k}} \left(e + \sum_{i=1}^k t_i u_i \right)$$

so

$$\begin{aligned} E_{F_0}(q, q^{-s}) &= \sum_{\mathbf{e} \in \mathbb{Z}^m \cap F_0} q^{(A+\gamma(F_0)) \cdot \mathbf{e} - (B \cdot \mathbf{e})s} \\ &= \left(\sum_{\mathbf{e} \in D_1} q^{(A+C_{F_0}) \cdot \mathbf{e} - (B \cdot \mathbf{e})s} \right) \prod_{i=1}^k \frac{1}{1 - q^{(A+C_{F_0}) \cdot \mathbf{u}_i - (B \cdot \mathbf{u}_i)s}}. \end{aligned}$$

Note that for each u_i , $i = 1, \dots, k$ we have $B \cdot u_i > 0$, so the sum converges for suitable s . Thus

$$\begin{aligned} E_{F_0}(q, q^{-s})|_{q \rightarrow q^{-1}} &= (-1)^{\dim F_0} \left(\sum_{\mathbf{e} \in D_0} q^{(A+C_{F_0}) \cdot \mathbf{e} - (B \cdot \mathbf{e})s} \right) \prod_{i=1}^k \frac{1}{1 - q^{(A+C_{F_0}) \cdot \mathbf{u}_i - (B \cdot \mathbf{u}_i)s}} \\ &= (-1)^{\dim F_0} \sum_{\mathbf{e} \in \mathbb{Z}^m \cap \overline{F_0}} q^{(A+\gamma(F_{\mathbf{e}})) \cdot \mathbf{e} - (B \cdot \mathbf{e})s} \\ &= (-1)^{\dim F_0} \sum_{F \leq F_0} E_F(q, q^{-s}). \end{aligned}$$

We now consider the general case. Since \mathcal{C} is a pointed cone, so is $\overline{F_0}$. By [Sta, Lemma 4.6.1], there exists a triangulation of $\overline{F_0}$, namely an expression for $\overline{F_0}$ as a union of simplicial cones closed under intersection and under taking faces. Comparing (2.4) (for F_0 simplicial) to Stanley's Lemma 4.6.13, and noting that $\overline{F_0} = \coprod_{F \leq F_0} F$, our result now follows from the proof of Stanley's Theorem 4.6.14. \square

We have

$$\begin{aligned} E_{\mathcal{C}_I}(q, q^{-s})|_{q \rightarrow q^{-1}} &= \sum_{F \in \mathcal{F}_I} E_F(q, q^{-s})|_{q \rightarrow q^{-1}} \\ &= \sum_{F' \in \mathcal{F}_I} \sum_{F \leq F'} (-1)^{\dim F'} E_F(q, q^{-s}) \quad (\text{by Lemma 2.5}) \\ &= \sum_{F \in \mathcal{F}} E_F(q, q^{-s}) \sum_{\substack{F' \in \mathcal{F}_I \\ F \leq F'}} (-1)^{\dim F'} \\ &= \sum_{F \in \mathcal{F}_{[m] \setminus I}} (-1)^m E_F(q, q^{-s}) \quad (\text{by Proposition 2.4}) \\ &= (-1)^m E_{\mathcal{C}_{[m] \setminus I}}(q, q^{-s}). \end{aligned}$$

This completes the proof of Theorem 2.3.

Now let W denote the Weyl group of an abstract root system Φ and let $\alpha_1, \dots, \alpha_l$ be a fundamental system of roots. Let Φ^+, Φ^- denote the sets of positive, negative roots respectively. We assume that $l < m$. In Section 5.2, where we derive our functional equations for the local zeta functions of \mathcal{G} , we will need to consider not just a single generating function as in Theorem 2.3, but a weighted sum over the Weyl group associated to \mathcal{G} of such generating functions. The cone \mathcal{C} will be defined by $m = l + d$ bounding hyperplanes, l of these corresponding to the fundamental roots, and d corresponding to the rank of the maximal central torus of \mathcal{G} . In our weighted sum each summand will be a generating function over a subset of \mathcal{C} with certain bounding hyperplanes (corresponding to a subset of the set of fundamental roots) excluded.

For each $w \in W$ set

$$(2.5) \quad I_w := \{i \in [l] \mid \alpha_i \in w(\Phi^-)\}.$$

The weighted sum is as follows.

Definition 2.6.

$$Z(q, q^{-s}) := \sum_{w \in W} q^{-\lambda(w)} E_{\mathcal{C}_{I_w}}(q, q^{-s}).$$

We now adapt the proof of [dSL, Theorem 5.9] to show that, under specified conditions, Theorem 2.3 implies that $Z(q, q^{-s})$ satisfies a functional equation. This reflects an interplay between a reciprocity of the generating function (Theorem 2.3) and a symmetry of the Weyl group. This interplay lies at the heart of the functional equation.

Corollary 2.7. *Suppose there exists a vector $\mathbf{a} \in \mathbb{Z}^m \cap \mathcal{C} \cap \bigcap_{i \in K} H_i$ such that for each $I \subseteq [l]$ we have*

$$(2.6) \quad \mathcal{C}_{[m] \setminus I} \cap \mathbb{Z}^m = \mathbf{a} + \mathcal{C}_{[l] \setminus I} \cap \mathbb{Z}^m.$$

Then, under the hypotheses of Theorem 2.3,

$$Z(q, q^{-s})|_{q \rightarrow q^{-1}} = (-1)^m q^{|\Phi^+| + (A + \gamma(F_{\mathbf{a}})) \cdot \mathbf{a} - (B \cdot \mathbf{a})s} Z(q, q^{-s}).$$

Proof.

$$\begin{aligned} Z(q, q^{-s})|_{q \rightarrow q^{-1}} &= (-1)^m \sum_{w \in W} q^{\lambda(w)} E_{\mathcal{C}_{[m] \setminus I_w}}(q, q^{-s}) && \text{(by Theorem 2.3)} \\ &= (-1)^m q^{(A + \gamma(F_{\mathbf{a}})) \cdot \mathbf{a} - (B \cdot \mathbf{a})s} \sum_{w \in W} q^{\lambda(w)} E_{\mathcal{C}_{[l] \setminus I_w}}(q, q^{-s}) \\ &= (-1)^m q^{(A + \gamma(F_{\mathbf{a}})) \cdot \mathbf{a} - (B \cdot \mathbf{a})s} \sum_{w \in W} q^{|\Phi^+| - \lambda(w w_0)} E_{\mathcal{C}_{I_{w w_0}}}(q, q^{-s}) \\ &= (-1)^m q^{|\Phi^+| + (A + \gamma(F_{\mathbf{a}})) \cdot \mathbf{a} - (B \cdot \mathbf{a})s} Z(q, q^{-s}), \end{aligned}$$

where $w_0 \in W$ is the longest element of the Weyl group. To see the second equality, note that by (2.6) it is enough to check that for every cell F there exists a cell F' such that $F \leq F'$ and $\mathbf{a} \in \overline{F'}$ and then to observe that γ may be replaced by a constant function on $\overline{F'}$. This follows from the fact that \mathcal{C} is a union of the chambers in \mathcal{F} , each of which contains $\mathcal{C} \cap \bigcap_{i \in K} H_i$ (we define a chamber to be the closure of a cell which is maximal with respect to the face relation). The third equality follows from the standard properties $\lambda(w) + \lambda(w w_0) = |\Phi^+|$ and $w_0(\Phi^-) = \Phi^+$ (see [Hum2, Section 1.8]). The latter gives

$$\begin{aligned} [l] \setminus I_w &= \{i \in [l] \mid \alpha_i \notin w(\Phi^-)\} \\ &= \{i \in [l] \mid \alpha_i \in w w_0(\Phi^-)\} \\ &= I_{w w_0}. \end{aligned}$$

□

3. INTEGRALS OVER REDUCTIVE GROUPS

Let \mathcal{G}_0 denote the connected component of \mathcal{G} . By [dSL, Lemma 4.1 and Proposition 2.1] we have that for almost all \mathfrak{p} , $Z_{\mathcal{G}, \rho, \mathfrak{p}}(s) = Z_{\mathcal{G}_0, \rho, \mathfrak{p}}(s)$. We therefore assume throughout that \mathcal{G} is connected. Fix a reductive subgroup G of \mathcal{G} such that $\mathcal{G} = N \rtimes G$, where N is the unipotent radical of \mathcal{G} . Fix a maximal torus T of G . We impose the following restriction.

Condition 3.1. *The maximal torus T of G splits over \mathfrak{K} ; that is, there exists a \mathfrak{K} -isomorphism $\phi : T \rightarrow \mathbb{G}_m^{\dim T}$.*

This implies in particular that T splits over $\mathfrak{K}_{\mathfrak{p}}$ for every prime \mathfrak{p} . One of the key observations made by du Sautoy and Lubotzky is that one is free to choose an equivalent \mathfrak{K} -rational representation. For a particular localization $\mathfrak{K}_{\mathfrak{p}}$, this may change the integral; however, they showed that for almost all \mathfrak{p} it will not. Specifically, they proved the following.

Lemma 3.2 ([dSL, Lemma 4.2]). *Let $\rho' : \mathcal{G} \rightarrow \mathrm{GL}_n$ be a \mathfrak{K} -rational representation equivalent to ρ ; that is, there exists $A \in \mathrm{GL}_n(\mathfrak{K})$ such that $\rho'(x) = A\rho(x)A^{-1}$ for all $x \in \mathcal{G}(\mathfrak{K})$. Then, for almost all primes \mathfrak{p} ,*

$$Z_{\mathcal{G}, \rho', \mathfrak{p}}(s) = Z_{\mathcal{G}, \rho, \mathfrak{p}}(s).$$

For our purposes we will require the representation ρ to satisfy number of properties. In view of Lemma 3.2 it will be sufficient to show that these properties are satisfied for some equivalent \mathfrak{K} -rational representation. This will ensure that we can pass to this equivalent representation for almost all primes \mathfrak{p} without changing the integral.

Definition 3.3. We call a \mathfrak{K} -rational representation ρ ‘good’ if it satisfies the following. Let the underlying ordered basis for the representation space V be $(u_i)_{i \in [n]}$. There exists a decomposition of $[n]$ as $[n] = \coprod_{i=1}^c I_i$ giving a decomposition of $V = \bigoplus_{i=1}^c U_i$, where U_i is the subspace of V spanned by $(u_j)_{j \in I_i}$, so that, putting $V_i = \bigoplus_{j=i}^c U_j$, we have

- (1) each V_i is \mathcal{G} -stable
- (2) each U_i is G -stable
- (3) N acts trivially on each section V_i/V_{i+1}
- (4) T acts diagonally with respect to the basis $(u_i)_{i \in [n]}$
- (5) each U_i is an irreducible subrepresentation of $\rho|_G$.

Proposition 3.4. *Given a faithful \mathfrak{K} -rational representation ρ of \mathcal{G} there exists an equivalent faithful \mathfrak{K} -rational representation ρ' which is good and satisfies, for almost all primes,*

$$Z_{\mathcal{G}, \rho', \mathfrak{p}}(s) = Z_{\mathcal{G}, \rho, \mathfrak{p}}(s).$$

Proof. By [dSL, Lemma 4.3], there exists a representation equivalent to ρ whose underlying ordered basis gives a decomposition $V = \bigoplus_{i=1}^c U'_i$ satisfying properties (1), (2) and (3) above. Each G -stable subspace U'_i can be decomposed further into irreducible components under the G -action. This gives a refinement of the original decomposition, say $V = \bigoplus_{i=1}^c U_i$, which is easily seen to satisfy properties (1), (2), (3) and (5). It remains to note that since T splits over \mathfrak{K} , there exists for each i a basis $(u_j)_{j \in I_i}$ for U_i on which T acts diagonally. The result now follows from Lemma 3.2. \square

We now assume that ρ is good and has the form described in Definition 3.3. We recall the following definitions from [dSL]. For each i let N_i denote the kernel of the action of N on V/V_{i+1} (so we obtain a normal series with $N_1 = N$ and $N_c = 1$). By identifying $\mathcal{U}_i := U_1 \oplus \dots \oplus U_i$ with V/V_{i+1} we obtain a faithful representation $\psi_i : N/N_i \rightarrow \mathrm{GL}(\mathcal{U}_i)$. This defines a unique representation $\varphi_i : \mathcal{G}/N_i \rightarrow \mathrm{GL}_n$

satisfying

$$\begin{aligned}\varphi_i(nN_i)(v) &= \psi_i(nN_i)(v) & \text{for all } n \in N, v \in \mathcal{U}_i \\ \varphi_i(nN_i)(v) &= v & \text{for all } n \in N, v \in V_{i+1} \\ \varphi_i(gN_i)(v) &= \rho(g)(v) & \text{for all } g \in G, v \in V.\end{aligned}$$

If $X/N_i \leq \mathcal{G}/N_i$ put $(X/N_i)^+ := \varphi_i^{-1}[\varphi_i((X/N_i)(\mathfrak{R}_{\mathfrak{p}})) \cap (M_n(\mathfrak{o}))]$. As in [dSL], we assume that the following condition holds:

Condition 3.5. *For each $i \in [c]$ and for each $\bar{g} \in (\mathcal{G}/N_i)^+$ there exists $g \in \mathcal{G}^+$ such that $gN_i = \bar{g}$.*

This appears as Assumption 2.3 in [dSL]. In the same paper, the authors state that the condition holds for almost all primes [dSL, Corollary 4.5]. As pointed out in Section 1, this is incorrect. We now explain how the integral may be reduced under Condition 3.5. Let μ_N denote the right Haar measure on N normalized such that $\mu_N(N(\mathfrak{o})) = 1$. Each section N_i/N_{i+1} admits a right Haar measure $\mu_{N_i/N_{i+1}}$ satisfying $\mu_{N_i/N_{i+1}}((N_i/N_{i+1})^+) = 1$. This can be used to define certain functions which describe an action of the reductive subgroup G on each section N_i/N_{i+1} :

Definition 3.6. For each $h \in G(\mathfrak{R}_{\mathfrak{p}})$ and $i \in [c-1]$ put

$$\theta_i(h) := \mu_{N_i/N_{i+1}}(\{n \in N_i/N_{i+1} \mid nh \in (\mathcal{G}/N_{i+1})^+\}).$$

Recall from (1.3) that $\theta(h) := \mu_N(\{n \in N \mid nh \in \mathcal{G}^+\})$. In [dSL], du Sautoy and Lubotzky reduced the integral $Z_{\mathcal{G}, \rho, \mathfrak{p}}(s)$ to an integral over the reductive subgroup G . It will be convenient for us to divide their result into two parts.

Proposition 3.7 (cf. (2.1) in the proof of [dSL, Theorem 2.2]). *Suppose that (\mathcal{G}, ρ) satisfies properties (1), (2) and (3) in Definition 3.3 and Condition 3.5 with respect to some suitable decomposition of the representation space. Then*

$$\theta(h) = \prod_{i=1}^{c-1} \theta_i(h).$$

Definition 3.8. Put

$$Z_{G, \rho, \theta, \mathfrak{p}}(s) := \int_{G^+} |\det \rho(h)|^s \theta(h) \mu_G(h).$$

Theorem 3.9 (cf. [dSL, Proof of Theorem 2.2]). *We have*

$$Z_{\mathcal{G}, \rho, \mathfrak{p}}(s) = Z_{G, \rho, \theta, \mathfrak{p}}(s).$$

Note that in contrast to Proposition 3.7, Theorem 3.9 is unconditional. In Section 4 when we come to analyze the integral using a \mathfrak{p} -adic Bruhat decomposition, we will need to understand how θ varies on Bruhat double cosets. Also, it will be crucial to express $\theta|_T$ in terms of characters of T . We now prove two results about the action of G on N which will enable us to deal with these issues.

Lemma 3.10. *For all $h \in G(\mathfrak{R}_{\mathfrak{p}})$ and $h_1, h_2 \in G(\mathfrak{o})$, $\theta(h) = \theta(h_1 h h_2)$.*

Proof. For $g \in G(\mathfrak{R}_{\mathfrak{p}})$ put $M_g = \{n \in N \mid ng \in \mathcal{G}^+\}$. It follows that $M_{h_1 h h_2}^{h_1} = M_h$. Consider a measure μ'_N on N given by $\mu'_N(A) = \mu_N(A^{h_1})$ for all measurable sets A . This defines a right Haar measure on N . By uniqueness of the Haar measure there exists $\lambda \in \mathbb{R}_{>0}$ such that $\mu'_N = \lambda \mu_N$. However, $\mu'_N(N(\mathfrak{o})) = \mu_N(N(\mathfrak{o})^{h_1}) =$

$\mu_N(N(\mathfrak{o}))=1$, so $\lambda = 1$. It follows that $\theta(h) = \mu_N(M_h) = \mu_N(M_{h_1 h h_2}) = \theta(h_1 h h_2)$. \square

Next we analyze $\theta|_T$ in the case that ρ is good (cf. Definition 3.3) and (\mathcal{G}, ρ) satisfies Condition 3.5. There exists a natural identification of N_i/N_{i+1} with a subspace of $U_{i+1}^{s_i}$, where $s_i = \sum_{j=1}^i \dim U_j$. This comes from considering the action of N_i/N_{i+1} on V/V_{i+2} . Since the induced action on V/V_{i+1} is trivial, N_i/N_{i+1} is identified with an additive algebraic subgroup of $U_{i+1}^{s_i} \cong \mathbb{G}_a^{s_i(s_{i+1}-s_i)}$. Note that $n \in (N_i/N_{i+1})^+$ if and only if its image in $U_{i+1}^{s_i}$ is in the \mathfrak{o} -span of the basis $(\{u_j\}_{j \in I_{i+1}})^{s_i}$. Now G acts on U_{i+1} , hence on $U_{i+1}^{s_i}$; in fact if $n \in N_i/N_{i+1}$, $g \in G$ and v is the image of n in $U_{i+1}^{s_i}$, then $ng \in (G/N_{i+1})^+$ if and only if $v.g$ is contained in the \mathfrak{o} -span of $(\{u_j\}_{j \in I_{i+1}})^{s_i}$. Thus the question of integrality is reduced to a question in linear algebra.

Definition 3.11. For each $\sigma \in \text{Sym}(n)$, put

$$T_\sigma(\mathfrak{K}_{\mathfrak{p}}) := \{t \in T(\mathfrak{K}_{\mathfrak{p}}) \mid v(\lambda_{\sigma(i)}(t)) \leq v(\lambda_{\sigma(j)}(t)) \text{ for all } 1 \leq i < j \leq n\},$$

where $\lambda_i(t)$ is the eigenvalue for the action of t on the basis element u_i .

Lemma 3.12. *Let ρ be a good representation of \mathcal{G} and suppose that, for almost all primes \mathfrak{p} , $(\mathcal{G}, \rho, \mathfrak{p})$ satisfies Condition 3.5. Define θ with respect to ρ as described above. Then for each $\sigma \in \text{Sym}(n)$, there exist non-negative integers $m_i(\sigma)$ ($i = 1, \dots, n$) such that, for almost all \mathfrak{p} , for all $t \in T_\sigma(\mathfrak{o})$,*

$$\theta(t) = |\lambda_1(t)|^{-m_1(\sigma)} \dots |\lambda_n(t)|^{-m_n(\sigma)}.$$

Proof. We fix a one-to-one correspondence $\tau \mapsto X_\tau$ between elements of $\text{Sym}(s_i(s_{i+1} - s_i))$ and orderings of the basis $X = (\{u_j\}_{j \in I_{i+1}})^{s_i}$ for $U_{i+1}^{s_i}(\mathfrak{K})$. Fix an ordered basis $(v_j)_{j \in [m]}$ for $N_i/N_{i+1}(\mathfrak{O})$, where $m := \dim N_i/N_{i+1}$. Given $\tau \in \text{Sym}(s_i(s_{i+1} - s_i))$, there exists an ordered basis $(w_j(\tau))_{j \in [m]}$ for $N_i/N_{i+1}(\mathfrak{K})$ with the following property:

(*) For all $j \in [m-1]$ and $k \in [s_i(s_{i+1} - s_i) - 1]$, if $w_j(\tau)$ has zero projection on the subspace generated by the first k basis elements of X_τ , then $w_{j+1}(\tau)$ has zero projection on the subspace generated by the first $k+1$ basis elements of X_τ .

For each τ we fix such an ordered basis for $N_i/N_{i+1}(\mathfrak{K})$ and denote it by Y_τ . Let $\Delta_i(\tau)$ denote the linear map $N_i/N_{i+1}(\mathfrak{K}) \rightarrow N_i/N_{i+1}(\mathfrak{K})$ given by $v_j \mapsto w_j(\tau)$ for $j = 1, \dots, m$. Note that if $t \in T$, t acts diagonally on the basis X for $U_{i+1}^{s_i}(\mathfrak{K})$, since ρ is good. Given σ , there exists $\tau \in \text{Sym}(s_i(s_{i+1} - s_i))$ such that, for all $t \in T_\sigma(\mathfrak{o})$, the valuations of the eigenvalues for the action of t on the ordered basis X_τ are in non-decreasing order. Fix some such suitable τ (there is some freedom here which need not concern us) and write the eigenvalues as $\nu_1(t), \dots, \nu_{s_i(s_{i+1}-s_i)}(t)$. Now fix $t \in T_\sigma(\mathfrak{o})$. Let Y_τ be the basis for $N_i/N_{i+1}(\mathfrak{K})$ chosen above and note that, for almost all \mathfrak{p} , the \mathfrak{o} -span of Y_τ is precisely $N_i/N_{i+1}(\mathfrak{o})$. This follows from the fact that the transformation $\Delta_i(\tau)$ defined above lies (for almost all \mathfrak{p}) in $\text{GL}_m(\mathfrak{o})$. By construction, Y_τ satisfies (*) with respect to X_τ . We may further assume that, in the expression for each element of Y_τ as a linear combination of the elements of the basis X_τ , every coefficient has valuation zero (this holds for almost all \mathfrak{p}). It now follows immediately that $\theta_i(t)$ has the form $\prod_{j=1}^{s_i(s_{i+1}-s_i)} |\nu_j(t)|^{-\delta_j}$ where each $\delta_j \in \{0, 1\}$. Note that for each $j \in [s_i(s_{i+1} - s_i)]$ there exists $k \in [s_{i+1}] \setminus [s_i]$ such

that $\nu_j(t) = \lambda_k(t)$, where $\lambda_k(t)$ is the eigenvalue for the action of t on u_k , so by Proposition 3.7 we obtain an expression of the required form by taking the product of the expressions obtained for each $\theta_i(t)$. It remains to note that this construction provides non-negative integers $m_j(\sigma)$ depending only on σ ; in particular they are independent of $t \in T_\sigma(\mathfrak{o})$ and of the localization $\mathfrak{K}_{\mathfrak{p}}$. \square

4. A COMBINATORIAL EXPRESSION FOR $Z_{G,\rho,\mathfrak{p}}(s)$

We begin by recalling the set-up of [dSL] in the reductive case (with some important distinctions). Let Φ be the root system of G relative to T (that is, those elements of $\text{Hom}(T, \mathbb{G}_m)$ which give rise to non-trivial weights for the adjoint action on the Lie algebra of G). As in Section 2, let $\alpha_1, \dots, \alpha_l$ be a set of fundamental roots for Φ and let W be the Weyl group. Let S be the maximal central torus of G and put $d := \dim S$. It is well-known that $G = S.G'$, where G' is the derived subgroup of G . Then G' is semisimple (see, for instance, [Bor, p. 10]). The root systems of G and G' are isomorphic, hence by [Hum1, 26.2, Corollary B (f)] the semisimple rank of G is l . It follows that the rank of G is $l + d$. To each root α there corresponds a minimal closed unipotent subgroup U_α of G such that conjugation by elements of T maps U_α into itself and there exists an isomorphism $\theta_\alpha : \mathbb{G}_a \rightarrow U_\alpha$ satisfying $t\theta_\alpha(x)t^{-1} = \theta_\alpha(\alpha(t)x)$ for all $x \in \mathbb{G}_a$, $t \in T$ (see [Hum1, 26.3, Theorem]). By Condition 3.1 we have a \mathfrak{K} -isomorphism $\phi : T \rightarrow \mathbb{G}_m^{l+d}$. We will need the following.

Condition 4.1. *The group G , the maximal torus T , and the isomorphisms $\phi : T \rightarrow (\mathbb{G}_m)^{l+d}$ and $\theta_\alpha : \mathbb{G}_a \rightarrow U_\alpha$ ($\alpha \in \Phi$) all have good reduction mod π .*

This was assumed also in [Igu]. As observed in [dSL, p. 74], it holds for almost all primes \mathfrak{p} . By good reduction of the maps we mean that reducing mod π gives induced maps $T(\mathfrak{o}/\mathfrak{p}) \rightarrow (\mathfrak{o}/\mathfrak{p})^{l+d}$ and $\mathfrak{o}/\mathfrak{p} \rightarrow U_\alpha(\mathfrak{o}/\mathfrak{p})$ which are isomorphisms. The finite Weyl group W of G is isomorphic to $N(T)/T$, where $N(T)$ is the normalizer of T in G . We define actions of W on $\text{Hom}(T, \mathbb{G}_m)$ and on $\text{Hom}(\mathbb{G}_m, T)$ as follows: if $w \in W$, $\alpha \in \text{Hom}(T, \mathbb{G}_m)$ and $\xi \in \text{Hom}(\mathbb{G}_m, T)$, put $(w\alpha)(t) = \alpha(w^{-1}(t))$ for all $t \in T$ and put $(w\xi)(\tau) = w(\xi(\tau))$ for all $\tau \in \mathbb{G}_m$, where the action of W on T is by conjugation. In particular, the former induces an action of W on Φ . A consequence of Condition 4.1 is that it is possible to take a coset representative g_w for each $w \in W = N(T)/T$ such that $g_w \in N(T)(\mathfrak{o})$ (see [Igu, p. 697]). We now put $\Xi := \text{Hom}(\mathbb{G}_m, T)$ and write $\mathcal{V} := \text{Hom}(T, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{R}$, $\mathcal{V}^* := \text{Hom}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that $\Xi \cong \mathbb{Z}^{l+d} \cong \text{Hom}(T, \mathbb{G}_m)$ since G has rank $l + d$.

For each $\alpha \in \text{Hom}(T, \mathbb{G}_m)$ and $\xi \in \text{Hom}(\mathbb{G}_m, T)$ let $\langle \alpha, \xi \rangle$ be the integer satisfying $\alpha(\xi(\tau)) = \tau^{\langle \alpha, \xi \rangle}$ for all $\tau \in \mathbb{G}_m$. This provides a pairing $\text{Hom}(T, \mathbb{G}_m) \times \text{Hom}(\mathbb{G}_m, T) \rightarrow \mathbb{Z}$ given by $(\alpha, \xi) \mapsto \langle \alpha, \xi \rangle$ which extends uniquely to a linear map $\mathcal{V} \times \mathcal{V}^* \rightarrow \mathbb{R}$. There exists a finite set Φ^* of coroots in Ξ which is invariant under the W -action on Ξ and satisfies the property that $\langle \alpha, \xi \rangle = \langle w\alpha, w\xi \rangle$ for all $w \in W$, $\alpha \in \Phi$, $\xi \in \Phi^*$. We define the affine Weyl group \mathcal{W} of G relative to T as $\mathcal{W} := W \ltimes \{t_\xi | \xi \in \Xi\}$ where $t_\xi : x \mapsto x + \xi$ is a translation on \mathcal{V}^* and the action is given by $(w_1 t_{\xi_1})(w_2 t_{\xi_2}) = w_1 w_2 t_{w_2^{-1}\xi_1 + \xi_2}$.

The chosen fundamental roots define sets of positive and negative roots which we write as Φ^+ and Φ^- respectively. Put

$$U^+ := \prod_{\alpha \in \Phi^+} U_\alpha$$

and similarly for U^- . We are now able to define the Iwahori subgroup. This is given as $\mathcal{B} := U^+(\pi\mathfrak{o})T(\mathfrak{o})U^-(\mathfrak{o})$. We have the following \mathfrak{p} -adic Bruhat Decomposition:

$$G(\mathfrak{K}_{\mathfrak{p}}) = \coprod_{wt_{\xi} \in \mathcal{W}} \mathcal{B}g_w\xi(\pi)\mathcal{B}$$

and

$$(4.1) \quad G(\mathfrak{o}) = \coprod_{w \in W} \mathcal{B}g_w\mathcal{B}.$$

([IM] and [Iwa, p. 74]). Define a length function λ on \mathcal{W} by

$$\begin{aligned} q^{\lambda(wt_{\xi})} &:= \text{card}(\mathcal{B}g_w\xi(\pi)\mathcal{B}/\mathcal{B}) \\ &= \mu_G(\mathcal{B}g_w\xi(\pi)\mathcal{B})/\mu_G(\mathcal{B}). \end{aligned}$$

The restriction of λ to the finite Weyl group agrees with the usual length function on W . For each $\xi \in \Xi$ there is a unique element $w_{\xi} \in W$ such that $\lambda(wt_{\xi})$, as a function of w , attains a minimum precisely at w_{ξ} and

$$(4.2) \quad \lambda(ww_{\xi}t_{\xi}) = \lambda(w) + \lambda(w_{\xi}t_{\xi}) \quad \text{for all } w \in W$$

[IM, p. 21]. As noted in [Igu, p. 704], it follows that

$$(4.3) \quad \lambda(w_{\xi}t_{\xi}) = \sum_{\alpha \in w_{\xi}^{-1}(\Phi^+)} \langle \alpha, \xi \rangle - \lambda(w_{\xi}).$$

Put

$$\begin{aligned} \Xi^+ &:= \{\xi \in \Xi \mid \xi(\pi) \in G^+\}, \\ \Xi_w &:= \{\xi \in \Xi \mid w_{\xi} = w\}, \\ \Xi_w^+ &:= \Xi^+ \cap \Xi_w. \end{aligned}$$

Proposition 4.2. *Put $\alpha_0 := \prod_{\alpha \in \Phi^+} \alpha$. Then for almost all primes \mathfrak{p} ,*

$$Z_{\mathcal{G}, \rho, \mathfrak{p}}(s) = \sum_{w \in W} q^{-\lambda(w)} \sum_{\xi \in w\Xi_w^+} q^{\langle \alpha_0, \xi \rangle} |\det \rho(\xi(\pi))|^s \theta(\xi(\pi)).$$

Note that this is essentially the same as [dSL, (5.4)]. In their setting, du Sautoy and Lubotzky assume that θ is (the \mathfrak{p} -adic absolute value of) a character of G , which allows them to express the product $|\det \rho(\xi(\pi))|^s \theta(\xi(\pi))$ in terms of a generator f for the (in their case one-dimensional) character group of G .

Proof. By Theorem 3.9 we have $Z_{\mathcal{G}, \rho, \mathfrak{p}}(s) = Z_{G, \rho, \theta, \mathfrak{p}}(s)$. We may therefore take Definition 3.8 as a starting point and follow the line of argument given in [dSL, pp. 76-77]. The \mathfrak{p} -adic Bruhat Decomposition gives

$$G^+ = \coprod_{\substack{wt_{\xi} \in \mathcal{W} \\ \xi \in \Xi^+}} \mathcal{B}g_w\xi(\pi)\mathcal{B}.$$

By construction, $\mathcal{B} \subseteq G(\mathfrak{o})$ and $g_w \in G(\mathfrak{o})$. It follows that each of $|\det \rho(h)|$ and $\theta(h)$ is constant on the double coset $\mathcal{B}g_w\xi(\pi)\mathcal{B}$, the latter by Lemma 3.10. In [dSL], this observation is combined with (4.1), (4.2) and (4.3) above to obtain the desired result. In their argument they use the fact that each of the maps $\xi \mapsto |\det \rho(\xi(\pi))|$

and $\xi \mapsto \theta(\xi(\pi))$ is constant on W -orbits in Ξ . It remains to check the latter of these in our setting: We have

$$\begin{aligned}\theta((w\xi)(\pi)) &= \theta(g_w^{-1}(\xi(\pi))g_w) \\ &= \theta(\xi(\pi)),\end{aligned}$$

by Lemma 3.10. \square

Next we need to use information about the weights of the representation $\rho|_G$. This will allow us to realize the sets $w\Xi_w^+$ as subsets of lattice points of a polyhedral cone so that the results of Section 2 will apply to the weighted sum in Proposition 4.2. Along the way, we will define a polyhedral cell complex that will allow us to incorporate the function θ into our analysis. Let ρ_1, \dots, ρ_r be the irreducible components of $\rho|_G$. For each irreducible component ρ_i , let $\omega_{i1}, \dots, \omega_{in_i} \in \text{Hom}(T, \mathbb{G}_m)$ be the weights of ρ_i (so $\sum_{i=1}^r n_i = n$) and let $\omega_i \in \text{Hom}(T, \mathbb{G}_m)$ be the dominant weight of the contragredient representation $g \mapsto {}^t\rho_i(g)^{-1}$. Then there exist $c_k(j, i) \in \mathbb{N}_0$ such that

$$(4.4) \quad \omega_{ij} = \omega_i^{-1} \prod_{k=1}^l \alpha_k^{c_k(j, i)}$$

for each $i \in [r]$ and $j \in [n_i]$ (see, for instance, [Hum1, 31.2, Proposition]). For convenience, for each $k \in [n]$ we write $\tilde{\omega}_k$ for the k^{th} weight in the ordering $(\omega_{11}, \dots, \omega_{1n_1}, \dots, \omega_{r1}, \dots, \omega_{rn_r})$. We will assume that the weights are ordered such that $\tilde{\omega}_k(t)$ is the eigenvalue for the action of $t \in T$ on the basis element u_k of V (see Definition 3.3). Fix a \mathbb{Z} -basis

$$(4.5) \quad \xi_1, \dots, \xi_{l+d}$$

for Ξ and let $f^* : \Xi \rightarrow \mathbb{Z}^{l+d}$ be the coordinate map relative to this basis. Let $f : \text{Hom}(T, \mathbb{G}_m) \rightarrow \mathbb{Z}^{l+d}$ be the coordinate map relative to the dual basis. Note that $\langle \alpha, \xi \rangle = f(\alpha) \cdot f^*(\xi)$ for all $\alpha \in \text{Hom}(T, \mathbb{G}_m)$, $\xi \in \Xi$, where the latter is the standard inner product on \mathbb{R}^{l+d} . For $\mathbf{v} \in \mathbb{Z}^{l+d}$, let $\underline{\xi}^{\mathbf{v}}$ denote $\xi_1^{v_1} \dots \xi_{l+d}^{v_{l+d}}$ (so $f^*(\underline{\xi}^{\mathbf{v}}) = \mathbf{v}$).

Igusa showed that

$$w\Xi_w = \{\xi \in \Xi \mid \langle \alpha_i, \xi \rangle \geq 0 \text{ for all } i; \langle \alpha_i, \xi \rangle > 0 \text{ if } \alpha_i \in w(\Phi^-)\}$$

[Igu, pp. 702-3]. From this and (4.4) du Sautoy and Lubotzky deduced that

$$(4.6) \quad w\Xi_w^+ = w\Xi_w \cap \{\xi \in \Xi \mid \langle \omega_j^{-1}, \xi \rangle \geq 0 \text{ for } j = 1, \dots, r\}.$$

[dSL, Lemma 5.4]. Put

$$\begin{aligned}B_i^{\geq} &:= \{\mathbf{e} \in \mathbb{R}^{l+d} \mid f(\alpha_i) \cdot \mathbf{e} \geq 0\} \quad \text{for } i = 1, \dots, l; \\ B_{l+j}^{\geq} &:= \{\mathbf{e} \in \mathbb{R}^{l+d} \mid f(\omega_j^{-1}) \cdot \mathbf{e} \geq 0\} \quad \text{for } j = 1, \dots, r.\end{aligned}$$

We define our polyhedral cone as

$$\mathcal{C} =: \bigcap_{i=1}^{l+r} B_i^{\geq}.$$

For each $i, j \in [n]$ with $i \neq j$, put

$$H_{ij}^{\leq} := \{\mathbf{e} \in \mathbb{R}^{l+d} \mid f(\tilde{\omega}_i \tilde{\omega}_j^{-1}) \cdot \mathbf{e} \leq 0\}.$$

For each $\sigma \in \text{Sym}(n)$ put

$$\begin{aligned}\Xi(\sigma) &:= \{\xi \in \Xi \mid \xi(\pi) \in T_\sigma(\mathfrak{R}_p)\}, \\ \Xi^+(\sigma) &:= \Xi^+ \cap \Xi(\sigma).\end{aligned}$$

(see Definition 3.11). Observe that

$$f^*(\Xi(\sigma)) = \bigcap_{1 \leq i < j \leq n} H_{\sigma(i)\sigma(j)}^{\leq}$$

and

$$\mathcal{C}_{I_w} = \mathcal{C} \cap \bigcap_{i \in I_w} B_i^> ,$$

with $I_w \subseteq [l]$ as defined in (2.5). We thus have

$$f^*(w\Xi_w^+) = \mathcal{C}_{I_w} \cap \mathbb{Z}^{l+d}.$$

We are now ready to define our polyhedral cell complex $(\mathcal{C}, \mathcal{F})$: take \mathcal{C} as above and H_{ij} ($1 \leq i < j \leq n$) as internal hyperplanes. By Lemma 3.12, for each $\sigma \in \text{Sym}(n)$ we can find non-negative integers $m_i(\sigma)$ such that

$$\theta(\underline{\xi}^e(\pi)) = q^{m_1(\sigma)\langle \tilde{\omega}_1, \underline{\xi}^e \rangle + \dots + m_n(\sigma)\langle \tilde{\omega}_n, \underline{\xi}^e \rangle}$$

for all $e \in f^*(\Xi^+(\sigma))$, and this equality holds for almost all primes p . Fix some total ordering on $\text{Sym}(n)$. To each cell F in the complex $(\mathcal{C}, \mathcal{F})$ associate the minimal element σ_F of $\text{Sym}(n)$ satisfying $F \subseteq f^*(\Xi^+(\sigma_F))$. Define a function $\gamma : \mathcal{F} \rightarrow \mathbb{Z}^{l+d}$ by

$$(4.7) \quad \gamma(F) := f \left(\prod_{i=1}^n (\tilde{\omega}_i)^{m_i(\sigma_F)} \right).$$

It follows that $\theta(\underline{\xi}^e(\pi)) = q^{\gamma(F_e) \cdot e}$ for all $e \in \Xi^+$ and that γ is piecewise constant on the complex (cf. Definition 2.1). We are ready to give our combinatorial expression. By Proposition 4.2, we have

Proposition 4.3. *For almost all primes p ,*

$$Z_{\mathcal{G}, \rho, p}(s) = \sum_{w \in W} q^{-\lambda(w)} E_{\mathcal{C}_{I_w}, A, B, \gamma}(q, q^{-s}),$$

where $A := f(\alpha_0)$, $B := f(\det \rho|_T)$, γ is the piecewise constant function $\mathcal{F} \rightarrow \mathbb{Z}^{l+d}$ defined in (4.7), and $m := l + d$ (cf. Definition 2.2).

5. PROOF OF THE MAIN THEOREM

5.1. Uniformity. Since we will need to apply Lemma 2.5, we first establish the following.

Lemma 5.1. *For all $w \in W$, $1 \neq \xi \in w\Xi_w^+$ and $F \in \mathcal{F}$, we have $\langle \alpha_0, \xi \rangle \geq 0$, $\langle \det \rho|_T, \xi \rangle > 0$ and $\langle \prod_{i=1}^n (\tilde{\omega}_i)^{m_i(\sigma_F)}, \xi \rangle \geq 0$. Furthermore, the cone \mathcal{C} is pointed.*

Proof. The first and third inequalities follow from (4.4) and (4.6). Note that $\det \rho|_T = \prod_{i=1}^n \tilde{\omega}_i$. Fix $w \in W$ and $\xi \in w\Xi_w^+$. By (4.4) and (4.6), $\langle \tilde{\omega}_i, \xi \rangle \geq 0$ for all $i \in [n]$. Thus $\langle \det \rho|_T, \xi \rangle \geq 0$. Suppose that $\langle \det \rho|_T, \xi \rangle = 0$. Then $\langle \tilde{\omega}_i, \xi \rangle = 0$ for all $i \in [n]$. However, the weights $\tilde{\omega}_i$ generate $\text{Hom}(T, \mathbb{G}_m)$, whence $\xi = 1$. The pointedness of \mathcal{C} follows directly from the fact that $\langle \det \rho|_T, \xi \rangle > 0$ for all $w \in W$, $1 \neq \xi \in w\Xi_w^+$. \square

Recall that $\mathcal{F}_{I_w} = \{F \in \mathcal{F} \mid F \subseteq \mathcal{C}_{I_w}\}$. By Proposition 4.3, we have

$$Z_{\mathcal{G}, \rho, \mathfrak{p}}(s) = \sum_{w \in W} q^{-\lambda(w)} \sum_{F \in \mathcal{F}_{I_w}} E_{F, A, B, \gamma}(q, q^{-s}).$$

Thus, by Lemma 5.1 and Lemma 2.5, we have that for almost all primes \mathfrak{p} , $Z_{\mathcal{G}, \rho, \mathfrak{p}}(s)$ is a sum of rational functions, hence is itself a rational function in q, q^{-s} , depending only on the group G and the representation ρ . This proves part (1) of Theorem 1.1.

5.2. The Functional Equation. In this section we complete the proof of Theorem 1.1 by applying the reciprocity results of Section 2. Suppose then that the number r of irreducible components of the representation ρ is equal to the dimension d of the maximal central torus S . In Section 4 we chose an arbitrary basis for Ξ (see (4.5)). We now specify this basis. Note that the group $\text{Hom}(T, \mathbb{G}_m)$ is generated by the weights, since ρ is faithful. However, by (4.4), the weights are generated by $\alpha_1, \dots, \alpha_l$ together with $\omega_1^{-1}, \dots, \omega_r^{-1}$. Since $r = d$ and $\text{Hom}(T, \mathbb{G}_m)$ has rank $l + d$ over \mathbb{Z} , this implies that $(\alpha_1, \dots, \alpha_l, \omega_1^{-1}, \dots, \omega_r^{-1})$ is a \mathbb{Z} -basis for $\text{Hom}(T, \mathbb{G}_m)$. We deduce that \mathcal{C} is simplicial. Write $\alpha_{l+i} := \omega_i^{-1}$ for $i = 1, \dots, d$. Let ξ_1, \dots, ξ_{l+d} be the dual basis to $\alpha_1, \dots, \alpha_{l+d}$; that is, the elements of Ξ having the property that $\langle \alpha_i, \xi_j \rangle = \delta_{ij}$ for all $i, j \in [l + d]$. We choose (ξ_i) and (α_i) as our (ordered) bases for Ξ and $\text{Hom}(T, \mathbb{G}_m)$ respectively and use them to define coordinate maps $f^* : \Xi \rightarrow \mathbb{Z}^{l+d}$ and $f : \text{Hom}(T, \mathbb{G}_m) \rightarrow \mathbb{Z}^{l+d}$ as described in Section 4. Note that $\mathcal{C} \cap \mathbb{Z}^{l+d} = \mathbb{N}_0^{l+d}$, so in fact \mathcal{C} is simple. Set $\mathbf{a} = f^*(\xi_{l+1} \dots \xi_{l+d}) = (\underbrace{0, \dots, 0}_l, \underbrace{1, \dots, 1}_d)$. It follows from the definitions and (4.4)

that

$$\mathbf{a} \in \mathbb{Z}^{l+d} \cap \mathcal{C} \cap \bigcap_{\substack{i, j \in [n] \\ i < j}} H_{ij}$$

and for each $I \subseteq [l]$ we have

$$\mathcal{C}_{[l+d] \setminus I} \cap \mathbb{Z}^{l+d} = \mathbf{a} + \mathcal{C}_{[l] \setminus I} \cap \mathbb{Z}^{l+d}.$$

By Proposition 4.3 and Corollary 2.7 we have

$$Z_{\mathcal{G}, \rho, \mathfrak{p}}(s)|_{q \rightarrow q^{-1}} = (-1)^{l+d} q^{|\Phi^+| + (f(\alpha_0) + \gamma(F_{\mathbf{a}})) \cdot \mathbf{a} - (f(\det \rho|_T) \cdot \mathbf{a})s} Z_{\mathcal{G}, \rho, \mathfrak{p}}(s).$$

By construction $f(\alpha_0) \cdot \mathbf{a} = \langle \prod_{\alpha \in \Phi^+} \alpha, \xi_{l+1} \dots \xi_{l+d} \rangle = 0$. Next, using (4.4) we have $f(\det \rho|_T) \cdot \mathbf{a} = \langle \prod_{i=1}^n \tilde{\omega}_i, \xi_{l+1} \dots \xi_{l+d} \rangle = n$. (Recall that n is the dimension of the representation ρ .) Thus our functional equation becomes

$$Z_{\mathcal{G}, \rho, \mathfrak{p}}(s)|_{q \rightarrow q^{-1}} = (-1)^{l+d} q^{|\Phi^+| + \gamma(F_{\mathbf{a}}) \cdot \mathbf{a} - ns} Z_{\mathcal{G}, \rho, \mathfrak{p}}(s).$$

By Lemma 5.1 we have that $\gamma(F_{\mathbf{a}}) \cdot \mathbf{a} \geq 0$. In particular, if \mathcal{G} is reductive then $\theta(h) = 1$ identically on $G = \mathcal{G}$ so $\gamma(F_{\mathbf{a}}) = 0$. This proves part (2) of Theorem 1.1.

As promised in Section 1, we now explain the error in the proof of [dSL, Theorem A]. The authors realize the zeta function in their setting as a generating function over a set I of lattices points of a cone. The existence of what they call a ‘dominating’ dominant weight guarantees that the cone is simplicial. In order to explicitly compute the zeta function, they further require the cone to be simple. To justify this, they state “Note that, since $\langle \omega_1^{-1}, \xi \rangle \in \mathbb{Z}$, $1/m \sum_{j=1}^l (b_j(1)/m_1 - c_j) e_j \in \mathbb{Z}$ ” (see [dSL, p. 82]). This inference is not valid, since the second expression is an integer if and only if $1/m_1 \langle \omega_1^{-1}, \xi \rangle$ is an integer

(cf. the expression for $\langle \omega_1^{-1}, \xi \rangle$ on [dSL, p. 80]). This is not true in general, since ω_1 is some dominant weight and there is no control over the integer m_1 defined on [dSL, p. 77]. If the ‘dominating’ dominant weight assumption [dSL, Assumption 5.5] is replaced by the more restrictive assumption that $|m_1| = 1$ (in particular, $r = 1$ is sufficient), their cone defining I becomes simple.

6. COUNTEREXAMPLES

Let T_d denote the d -dimensional split torus ($d \geq 2$). We define an infinite family of representations of T_d whose associated local zeta functions do not satisfy functional equations. In these examples, $r = 2d - 1$, illustrating that the condition $r = d$ in Theorem 1.1 cannot be dropped.

Proposition 6.1. *Let $G = T_d := \mathbb{G}_m^d$, where $d \geq 2$. For each integer $k \geq 3$ define a faithful \mathbb{Q} -rational representation $\rho_{d,k} : G \rightarrow \mathrm{GL}_{2d-1}$ by $\rho_{d,k}(x_1, \dots, x_d) = \mathrm{diag}(x_1, \dots, x_d, x_1^k x_d^{-1}, \dots, x_{d-1}^k x_d^{-1})$. For all primes p ,*

$$Z_{G, \rho_{d,k}, p}(s) = \int_{G^+} |\det \rho_{d,k}(g)|^s \mu_G(g)$$

does not satisfy a functional equation.

Proof. Fix d and k , writing $\rho = \rho_{d,k}$ and $G = G(\mathbb{Q}_p)$. For $\mathbf{y} \in \mathbb{Z}^d$ put

$$G_{\mathbf{y}} = \{\mathbf{x} \in G \mid v(x_i) = y_i \text{ for } i = 1, \dots, d\}.$$

We have

$$G_{\mathbf{0}} = \{g \in G \mid \rho(g) \in \mathrm{GL}_{2d-1}(\mathbb{Z}_p)\} = G(\mathbb{Z}_p),$$

and for all $\mathbf{y} \in \mathbb{Z}^d$,

$$G_{\mathbf{y}} = G_{\mathbf{0}} \cdot (p^{y_1}, \dots, p^{y_d}).$$

This implies that $\mu_G(G_{\mathbf{y}}) = \mu_G(G_{\mathbf{0}}) = 1$. If $\mathbf{x} \in G$, put $y_i = v(x_i)$ for $i = 1, \dots, d$. Put

$$\mathcal{C} = \{\mathbf{u} \in \mathbb{R}_{\geq 0}^d \mid ku_i - u_d \geq 0 \text{ for } i = 1, \dots, d-1\}.$$

Then $\mathbf{x} \in G^+ \iff \mathbf{y} \in \mathcal{C} \cap \mathbb{Z}^d$. Setting $X_i = p^{-(k+1)s}$ for $i = 1, \dots, d-1$ and $X_d = p^{(d-2)s}$ we have

$$\begin{aligned} Z_{G, \rho, p}(s) &= \int_{G^+} |\det \rho(g)|^s \mu_G(g) \\ &= \sum_{\mathbf{y} \in \mathcal{C} \cap \mathbb{Z}^d} p^{(-(k+1)(y_1 + \dots + y_{d-1}) + (d-2)y_d)s} \mu_G(G_{\mathbf{y}}) \\ &= \sum_{\mathbf{y} \in \mathcal{C} \cap \mathbb{Z}^d} X_1^{y_1} \dots X_d^{y_d}. \end{aligned}$$

Let $\{e_1, \dots, e_d\}$ be the standard basis for \mathbb{R}^d . Put $f_i = e_i$ for $i = 1, \dots, d-1$ and $f_d = ke_d + \sum_{j=1}^{d-1} e_j$. It is straightforward to check that $\mathcal{C} = \mathrm{span}_{\mathbb{R}_{\geq 0}^d} \{f_1, \dots, f_d\}$. Put $D_0 = \{0\} \cup \{je_d + \sum_{i=1}^{d-1} e_i \mid j = 1, \dots, k-1\}$. Another routine check shows that

$$\mathcal{C} \cap \mathbb{Z}^d = \coprod_{\mathbf{u} \in D_0} (\mathbf{u} + \mathrm{span}_{\mathbb{N}_0} \{f_1, \dots, f_d\}).$$

It follows that

$$Z_{G,\rho,p}(s) = \frac{1 + X_1 \dots X_d(1 + X_d + \dots + X_d^{k-2})}{(1 - X_1)(1 - X_2) \dots (1 - X_{d-1})(1 - X_1 \dots X_{d-1} X_d^k)},$$

hence

$$Z_{G,\rho,p}(s)|_{x_i \rightarrow x_i^{-1}} = (-1)^d \frac{X_1 \dots X_d(1 + X_d + \dots + X_d^{k-2} + X_1 \dots X_{d-1} X_d^{k-1})}{(1 - X_1)(1 - X_2) \dots (1 - X_{d-1})(1 - X_1 \dots X_{d-1} X_d^k)}.$$

Suppose that $Z_{G,\rho,p}(s)$ satisfies a functional equation of the form

$$Z_{G,\rho,p}(s)|_{p \rightarrow p^{-1}} = (-1)^m p^{a+bs} Z_{G,\rho,p}(s).$$

Then

$$\begin{aligned} & (-1)^m p^{a+bs} (1 + X_1 \dots X_d(1 + X_d + \dots + X_d^{k-2})) \\ &= (-1)^d X_1 \dots X_d(1 + X_d + \dots + X_d^{k-2} + X_1 \dots X_{d-1} X_d^{k-2}). \end{aligned}$$

If $d = 2$, comparing highest powers of p^{-s} immediately leads to a contradiction. If $d > 2$, comparing lowest and next-to-lowest powers of p^{-s} gives $p^{a+bs} = p^{-(d-2)s}$ and $X_1 \dots X_{d-1} X_d^k = 1$, hence $d + k - 1 = 0$, which is impossible. \square

It is easy to see that there are also families of representations of T_d with $r > d$ whose associated zeta functions do satisfy a functional equation. If the associated cone \mathcal{C} is simplicial, the existence of a functional equation in fact depends on the configuration of the lattice points in \mathcal{C} . Recalling the definitions of the sets D_0 and D_1 used in the proof of Lemma 2.5, we note that the generating function satisfies a functional equation if and only if D_0 maps onto D_1 under a translation. This is a highly restrictive condition on the cone.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF CAPE TOWN,
PRIVATE BAG, RONDEBOSCH 7701, CAPE TOWN, SOUTH AFRICA
E-mail address: mark.berman@uct.ac.za

REFERENCES

- [Ber] M.N. Berman, *Proisomorphic zeta functions of groups*, D.Phil. thesis, Oxford, 2005.
- [BG] R.N. Bryant and J.R.J. Groves, Algebraic groups of automorphisms of nilpotent groups and Lie algebras, *J. London Math. Soc.* **33** (1986), 453–466.
- [Bor] A. Borel, *Linear algebraic groups*, Algebraic groups and discontinuous subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), 1966, pp. 3–19.
- [BT1] F. Bruhat and J. Tits, Groupes réductifs sur un corps local: I. Données radicielles valuées, *Inst. hautes Études sci. publ. math.* **41** (1972), 5–251.
- [BT2] ———, Groupes réductifs sur un corps local. II. Schémas en groupes. existence d’une donnée radicielle valuée., *Inst. hautes Études sci. publ. math.* **60** (1984), 197–376.
- [dS] M.P.F. du Sautoy, A nilpotent group and its elliptic curve: non-uniformity of local zeta functions of groups, *Israel J. Math.* **126** (2001), 269–288.
- [dSG] M.P.F. du Sautoy and F.J. Grunewald, Analytic properties of zeta functions and subgroup growth, *Ann. of Math.* **152** (2000), 793–833.
- [dSL] M.P.F. du Sautoy and A. Lubotzky, Functional equations and uniformity for local zeta functions of nilpotent groups, *Amer. J. Math.* **118** (1996), no. 1, 39–90.
- [dSW] M.P.F. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, Lecture Notes in Mathematics, vol. 1925, Springer.
- [GSS] F.J. Grunewald, D. Segal, and G.C. Smith, Subgroups of finite index in nilpotent groups, *Invent. Math.* **93** (1988), 185–223.
- [Hey] K. Hey, *Analytische Zahlentheorie in Systemen hyperkomplexer Zahlen*, Ph.D. thesis, Hamburg, 1929.

- [Hum1] J.E. Humphreys, *Linear Algebraic Groups, graduate texts in mathematics*, Springer-Verlag, Graduate Texts in Mathematics 21, 1981.
- [Hum2] ———, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, 1990.
- [Igu] J.-I. Igusa, Universal p -adic zeta functions and their functional equations, *Amer. J. Math.* **111** (1989), 671–716.
- [IM] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p -adic chevalley groups, *Inst. Hautes Études Sci. Publ. Math.* **25** (1965), 5–48.
- [Iwa] N. Iwahori, Generalized Tits System (Bruhat Decomposition) on \mathfrak{p} -Adic Semisimple Groups, *Proc. Symposia Pure Math.* **9** (1966), 71–83.
- [KV] B. Klopsch and C. Voll, Igusa-type functions associated to finite-formed spaces and their functional equations, *Transactions of the American Mathematical Society* **361** (2009), 4405–4436.
- [Mac] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Clarendon Press, Oxford, 1995.
- [Mar] R. Martin, A counterexample in the theory of local zeta functions, *Experiment. Math.* **4** (1995), 299–305.
- [Paa] P.M. Paaianen, Structure and functional equations of zeta functions of class two nilpotent groups, *arXiv:math/0612511v2*.
- [Sat] I. Satake, Theory of spherical functions on reductive algebraic groups over \mathfrak{p} -adic fields, *Inst. Hautes Études Sci. Publ. Math.* **18** (1963), 5–70.
- [Sta] R. Stanley, *Enumerative combinatorics*, vol. 1, Cambridge University Press, 1997.
- [Tam] T. Tamagawa, On the ζ -function of a division algebra, *Ann. of Math.* **77** (1963), 387–405.
- [Vol] C. Voll, Functional equations for zeta functions of groups and rings, *arXiv:math/0612511v2*, to appear in *Ann. of Math.*
- [Wei] A. Weil, *Adèles and Algebraic Groups*, Birkhäuser-Verlag, 1962.